Modal Logic (part 1)

L. Yohanes Stefanus

Faculty of Computer Science
University of Indonesia, Jakarta/Depok

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Logic in Computer Science

- The goal of logic is to develop languages to model the situations we encounter, in such a way that we can reason about them formally.
- Reasoning about situations means constructing arguments about them; it's doneformally means that it's done in such a way that the arguments are valid and can be defended mathematically, or executed on a computer.

Modes of Truth

- In propositional or predicate logic, formulas are either true or false, in any model.
- This is inadequate. In many applications, it is often necessary to distinguish among various modes of truth, such as *necessarily true, known to be true, believed to be true, true in the future*.
- For example, the sentence "The cube root of 125 is 5." is *true*, also *necessarily true* and *true in the future*. However, it does not enjoy all modes of truth. It may not be *known to be true* by some people (children, for example); it may not be *believed to be true* by others (if they are mistaken).

Modes of Truth

- Modal logic adds unary connectives to express one, or more, of those different modes of truth.
- The simplest modal logics deal only with one concept – such as knowledge, necessity, or time.
- More sophisticated modal logics have connectives for expressing several modes of truth in the same logic.

Main Applications of Modal Logic

- Natural language processing/understanding
- Intelligent multi-agent systems
- Real-time systems
- Semantic networks

Syntax

- The language of basic modal logic is that of propositional logic with two extra unary connectives, $\Box$ and $\Diamond$.
- Atoms = $\{p, q, r, \ldots, p_1, p_2, p_3, \ldots\}$ denotes the set of atomic formulas in propositional logic.

Syntax: Well-Formed Formula (wff)

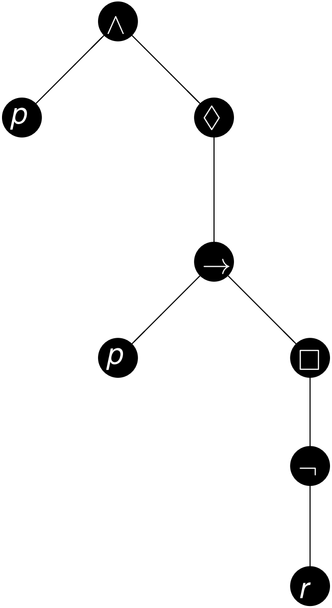
Definition

The well-formed formulas of basic modal logic ϕ are defined by the following Backus Naur form:

$$\begin{aligned} \phi \quad ::= \quad & \bot \mid \top \mid p \mid (\neg\phi) \mid (\phi \land \phi) \mid (\phi \lor \phi) \mid (\phi \to \phi) \mid \\ & (\phi \leftrightarrow \phi) \mid (\Box\phi) \mid (\Diamond\phi) \end{aligned} \tag{1}$$

where p is any atomic formula.

- **Example**: $(p \land \Diamond(p \to \Box\neg r))$


∧
p
◊
→
p
□
¬
r

Validity

- Which modal formulae are valid?
- The method of defining validity for modal wff which are widely applicable is based on the following ideas:
  - **(a)** Whereas determining the truth-value of a non-modal proposition involves only a consideration of how things actually are, determining the truth-value of a proposition of the form $\Box p$ ('necessarily p') or $\Diamond p$ ('possibly p') involves a consideration of how things might have been, of the nature of conceivable worlds alternative to the actual one.

Validity

(b) For each conceivable world there is a range of worlds which are possible relative to that one. This reflects the idea we sometimes express by saying that if things were different a new range of possibilities might be opened up, so that things that are not even possible as things stand might be possible then.

(c) In any given conceivable world, $\Diamond p$ counts as true iff p itself would be true in at least one world which is possible relative to that one, and $\Box p$ counts as true iff p itself would be true in every such world.

Modal Logic Game

Setting-up:

- There can be any number of players, provided that there is at least one.
- The players are seated in some way which determines precisely which players, if any, each player is to be able to see during the course of the game. In this context, being able to see someone means no more than taking note of that person's responses.
- There are no restrictions whatsoever on what "seeing arrangement" among the players may be made: thus we may decide that no one is to be able to see anyone at all, or at the opposite extreme that everyone can see everyone, or some players shall be able to see themselves while others shall not, or if player A can see player B, B may or may not be allowed to see A, etc.

Modal Logic Game

Setting-up:

- Before the game begins, each player is provided with a sheet of paper on which we have previously written a number of letters of the alphabet (p, q, r, etc.).
- We shall call the set of players together with the specification of who is to be able to see whom, a seating arrangement, and this together with the players' sheets a setting.
- The game proceeds by calling, to the whole set of players at once, any wff of modal logic we choose, provided that its well-formed parts, beginning with the variables, are called first.

Modal Logic Game

The instructions for each player:

(1) If a single letter (variable) is called, raise your hand if that letter is on the sheet; keep it down if it is not.

(2) If $\neg\alpha$ is called (where α is a wff) raise your hand if you kept it down when α was called; keep it down if you raised it when α was called.

(3) If $(\alpha \vee \beta)$ is called, raise your hand if you raised it for α or for β; keep it down if you kept it down for both α and β.

(4) If $(\alpha \wedge \beta)$ is called, raise your hand if you raised it for α and for β; keep it down if you kept it down for α or β.

(5) If $(\alpha \rightarrow \beta)$ is called, raise your hand if you kept it down for α or raised it for β; keep it down if you raised it for α and kept it down for β.

Modal Logic Game

The instructions for each player:

(6) If $\Box\alpha$ is called (where α is a wff), raise your hand if every player you can see raised his or her hand when α was called; otherwise keep your hand down.

(7) If $\Diamond\alpha$ is called (where α is a wff), raise your hand if at least one of the players you can see raised his or her hand when α was called; otherwise keep your hand down.

Modal Logic Game

How should we use this game to define validity in modal logic?

- A wff α is valid in a given seating arrangement iff in that seating arrangement all the players would raise their hands for α, no matter what sheets were distributed to them.
- This means there will be as many different kinds of validity for modal formulae as there are different seating arrangements.
- The possibility of having different kinds of seating arrangements is part of what gives modal logic its richness.

Modal Logic Game

Example 1:

- Take wff $\Box p \rightarrow p$ with a seating arrangement in which there are only two players, A and B, and both can see themselves and each other. The letter p is on A's sheet but not on B's.
- Is the wff valid in this seating arrangement?

Modal Logic Game

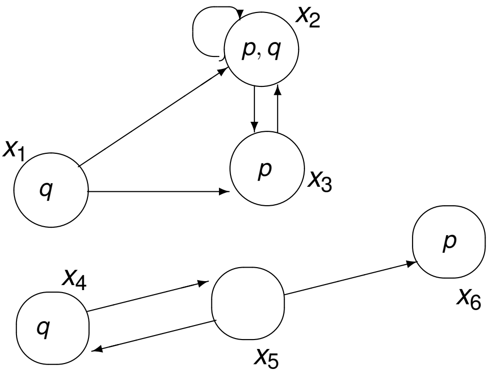
Example 2:

- Take wff $\Box p \rightarrow p$ with a seating arrangement just like in the previous example except that A cannot see himself or herself. The letter p is on B's sheet but not on A's.
- Is the wff valid in this seating arrangement?

Modal Logic Game

Example 3:

- Take wff $\Box(p \to q) \to (\Box p \to \Box q)$ with the seating arrangement shown below in which there are six players $\{x_1, x_2, x_3, x_4, x_5, x_6\}$. The arrow means "can see".
- Is the wff valid in this seating arrangement?



The Meaning of the Modal Logic Game

- In any setting the players represent conceivable possible worlds.
- The players each player is allowed to see represent the worlds which are possible relative to the world which that player represents.
- The letters on a player's sheet represent the propositions that are true in that world.
- Raising a hand and keeping it down represent respectively truth and falsity in the world the player represents.

Semantics

- In propositional logic, a model is simply an assignment of truth values to each of the atomic formulas which are present. Such models are also called valuations.
- However, this notion of model is inadequate for modal logic, since we want to distinguish between different modes, or degrees, of truth.

Definition

A model $\mathcal{M}$ of basic modal logic is specified by three things:

1. A set W, whose elements are called worlds;
2. A relation R on W (that is, $R \subseteq W \times W$), called the accessibility relation;
3. A function $L : W \to \mathcal{P}(\text{Atoms})$, called the labelling function.

- These models are often called Kripke models, in honour of S. Kripke.
- Intuitively, $w \in W$ stands for a possible world and $R(w, w')$ means that w' is a world accessible from world w. The actual nature of that relationship depends on what we intend to model.
- Elements $w \in W$ are also called points, nodes, state descriptions, doxastic states (i.e., belief, knowledge), times, instants, situations.

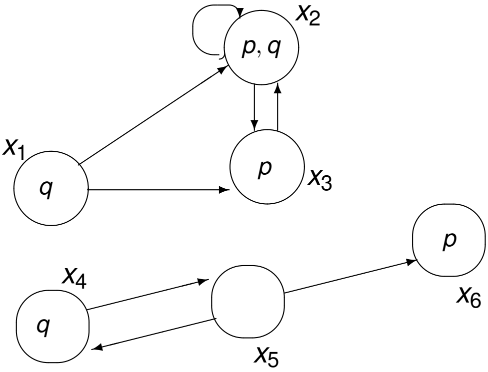
Example

- Let W be $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the relation R be defined as follows: $R(x_1, x_2)$, $R(x_1, x_3)$, $R(x_2, x_2)$, $R(x_2, x_3)$, $R(x_3, x_2)$, $R(x_4, x_5)$, $R(x_5, x_4)$, $R(x_5, x_6)$; and no other pairs are related by R. Further, let the labelling function behave as follows:

x	x_1	x_2	x_3	x_4	x_5	x_6
$L(x)$	$\{q\}$	$\{p, q\}$	$\{p\}$	$\{q\}$	$\emptyset$	$\{p\}$

- The Kripke model can be depicted by a graphical notation. The set W is drawn as a set of circles, with arrows between them showing the relation R. Within each circle is the value of the labelling function in that world.

A Kripke model in graphical notation



Possible Worlds Semantics

Definition

Let $\mathcal{M} = (W, R, L)$ be a model of basic modal logic. Suppose $x \in W$ and ϕ is a formula defined by (1) on Slide 9. We define when formula ϕ is true in the world x via a satisfaction relation $x \Vdash \phi$ by structural induction on ϕ:

$$
\begin{array}{lll}
x & \Vdash & \top \\
x & \nVdash & \bot \\
x & \Vdash & p \quad \text{iff} \quad p \in L(x) \\
x & \Vdash & \neg\phi \quad \text{iff} \quad x \nVdash \phi \\
x & \Vdash & \phi \wedge \psi \quad \text{iff} \quad x \Vdash \phi \quad \text{and} \quad x \Vdash \psi \\
x & \Vdash & \phi \vee \psi \quad \text{iff} \quad x \Vdash \phi \quad \text{or} \quad x \Vdash \psi
\end{array}
$$

Possible Worlds Semantics

Definition

(cont.)

$$x \Vdash \phi \to \psi \quad \text{iff} \quad x \Vdash \psi, \quad \text{whenever we have} \quad x \Vdash \phi$$

$$x \Vdash \phi \leftrightarrow \psi \quad \text{iff} \quad (x \Vdash \phi \quad \text{iff} \quad x \Vdash \psi)$$

$$x \Vdash \Box\psi \quad \text{iff, for each } y \in W \text{ with } R(x, y), \text{ we have } y \Vdash \psi$$

$$x \Vdash \Diamond\psi \quad \text{iff there is a } y \in W \text{ such that } R(x, y) \text{ and } y \Vdash \psi.$$

When $x \Vdash \phi$ holds, we say 'x satisfies ϕ', or 'ϕ is true in world x'. We write $\mathcal{M}, x \Vdash \phi$ if we want to stress that $x \Vdash \phi$ holds in the model $\mathcal{M}$.

- The first two clauses say that $\top$ is always true, while $\bot$ is always false.
- $L(x)$ is the set of all the atomic formulas that are true at x.
- The clauses for the boolean connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$) are straightforward: they mean that we apply the usual truth-table semantics of these connectives in the current world x.
- For $\Box\phi$ to be true at x, it is required that ϕ be true in all the worlds accessible from x.
- For $\Diamond\phi$ to be true at x, it is required that there is at least one accessible world from x in which ϕ is true.

Example

For the Kripke model on Slide 25, we have:

- $x_1 \Vdash q$, because $q \in L(x_1)$.
- $x_1 \Vdash \Diamond q$, because there is a world accessible from x_1 (namely, x_2) which satisfies q.
- $x_1 \nVdash \Box q$. This is because $x_1 \Vdash \Box q$ says that all worlds accessible from x_1 (i.e. x_2 and x_3) satisfy q; but x_3 does not.
- The worlds accessible from x_5 are x_4 and x_6. Since $x_4 \nVdash p$, we have $x_5 \nVdash \Box p$; and since $x_6 \nVdash q$, we have $x_5 \nVdash \Box q$. Therefore, $x_5 \nVdash \Box p \vee \Box q$. However, $x_5 \Vdash \Box(p \vee q)$ holds because, in x_4 and x_6, we find p or q.
- The worlds which satisfy $\Box p \to p$ are $x_2,\ x_3,\ x_4,\ x_5$ and x_6. x_1 cannot satisfy $\Box p \to p$ since it satisfies $\Box p$ but not p itself.

Example (cont.)

- Worlds like x_6 that have no world accessible from them deserve special attention in modal logic. $x_6 \not\Vdash \Diamond\phi$, no matter what ϕ is, because there is no accessible world from x_6. In particular, $x_6 \not\Vdash \Diamond\top$. However, we have $x_6 \Vdash \top$.
  $x \Vdash \Diamond\top$ holds iff x has at least one accessible world.
- No matter what ϕ is, $x_6 \Vdash \Box\phi$. This is because $x_6 \Vdash \Box\phi$ says that ϕ is true in all worlds accessible from x_6, and there are no such worlds, so ϕ is vacuously true in all of them (there is simply nothing to check). In particular, $x_6 \Vdash \Box\bot$.
  $x \Vdash \Box\bot$ holds iff x has no accessible worlds.

Formula Schemes

- It is sometimes useful to talk about a whole family of formulas which have the same 'shape'; these are called formula schemes.
- Any formula which has the same shape of a certain formula scheme is called an instance of the scheme.
- For example,
  - $p \to \Box\Diamond p$
  - $q \to \Box\Diamond q$
  - $(p \wedge \Diamond q) \to \Box\Diamond(p \wedge \Diamond q)$

  are all instances of the scheme $\phi \to \Box\Diamond\phi$.
- Semantically, a scheme can be thought of as the conjunction of all its instances.
- We say that a world/model satisfies a scheme if it satisfies all its instances.

Equivalences between modal formulas

Definition

1. We say that a formula ϕ of basic modal logic semantically entails a formula ψ of basic modal logic if, in any world x of any model $\mathcal{M} = (W, R, L)$, we have $x \Vdash \psi$ whenever $x \Vdash \phi$. In that case, we say that $\phi \models \psi$ holds.
2. We say that ϕ and ψ are semantically equivalent if $\phi \models \psi$ and $\psi \models \phi$ hold. We denote this by $\phi \equiv \psi$.
3. We say that a set of formulas Γ of basic modal logic semantically entails a formula ψ of basic modal logic if, in any world x of any model $\mathcal{M} = (W, R, L)$, we have $x \Vdash \psi$ whenever $x \Vdash \phi$ for all $\phi \in \Gamma$. In that case, we say that $\Gamma \models \psi$ holds.

- Any equivalence in propositional logic is also an equivalence in modal logic.
- De Morgan rules: $\neg\Box\phi \equiv \Diamond\neg\phi$ and $\neg\Diamond\phi \equiv \Box\neg\phi$.
- $\Box$ distributes over $\wedge$: $\Box(\phi \wedge \psi) \equiv \Box\phi \wedge \Box\psi$.
- $\Diamond$ distributes over $\vee$: $\Diamond(\phi \vee \psi) \equiv \Diamond\phi \vee \Diamond\psi$.
- $\Box\top \equiv \top$.
- $\Diamond\bot \equiv \bot$.
- $\Diamond\top \equiv \Box p \rightarrow \Diamond p$.

Valid Formulas

Definition

A formula ϕ of basic modal logic is said to be valid if it is true in every world of every model, i.e. iff $\models \phi$ holds.

Any propositional tautology is a valid formula and so is any substitution instance of it. A substitution instance of a formula is the result of uniformly substituting the atoms of the formula by other formulas.

Example

$\neg\Box\phi \leftrightarrow \Diamond\neg\phi$ is a valid formula.

Proof:

Suppose x is a world in a model $\mathcal{M} = (W, R, L)$. We have to show $x \Vdash \neg\Box\phi \leftrightarrow \Diamond\neg\phi$, i.e. we have to show that $x \Vdash \neg\Box\phi$ iff $x \Vdash \Diamond\neg\phi$.

Using Definition on Slide 26, we have

$x \Vdash \neg\Box\phi$

iff it is not the case that $x \Vdash \Box\phi$

iff it is not the case that, for all y such that $R(x, y)$, $y \Vdash \phi$

iff there is some y such that $R(x, y)$ and not $y \Vdash \phi$

iff there is some y such that $R(x, y)$ and $y \Vdash \neg\phi$

iff $x \Vdash \Diamond\neg\phi$.

Other interesting valid formulas in basic modal logic:

- Formula K: $\Box(\phi \to \psi) \wedge \Box\phi \to \Box\psi$.
  This formula is sometimes written in the equivalent form:
  $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.
- $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$.
- $\Diamond(\phi \vee \psi) \leftrightarrow \Diamond\phi \vee \Diamond\psi$.

Logic Engineering

- The basic modal logic is quite general and can be refined in various ways to give us the properties appropriate for the intended applications.
- Logic Engineering is the subject which tries to engineer logics to fit new applications. It draws on all branches of Logic, Computer Science and Mathematics.
- We will consider how to re-engineer basic modal logic to fit the readings of $\Box\phi$ given on Slide 38.
- As $\Diamond$ is equivalent to $\neg\Box\neg$, we can find out what the corresponding readings of $\Diamond\phi$.

The Readings of $\Diamond$ Corresponding to Each Reading of $\Box\phi$

$\Box\phi$	$\Diamond\phi$
It is necessarily true that ϕ	It is possibly true that ϕ
Agent Q believes that ϕ	ϕ is consistent with Q's beliefs
Agent Q knows that ϕ	For all Q knows, ϕ
After any execution of program P, ϕ holds	After some execution of P, ϕ holds

- With the reading it is necessarily true that ϕ for $\Box\phi$, we can derive the reading for $\Diamond\phi$ as follows:

  it is not necessarily true that not ϕ
  = it is possible that not not ϕ
  = it is possible that ϕ.

- With the reading agent Q knows that ϕ for $\Box\phi$, we can derive the reading for $\Diamond\phi$ as follows:

  agent Q does not know that not ϕ
  = as far as Q's knowledge is concerned,
  ϕ could be the case
  = ϕ is consistent with what agent Q knows
  = for all agent Q knows, ϕ.

- Part of the job of logic engineering is to determine what formula schemes should be valid and to craft the logic in such a way that precisely those ones are valid.
- If we are to build a logic capturing the concept of knowledge, what formula schemes should be valid?

- $\Box\phi \to \phi$
  Anything which is known must also be true.
- $\Box\phi \to \Box\Box\phi$
  This is called positive introspection. If agent Q knows something, she knows that she knows it.
- $\Diamond\phi \to \Box\Diamond\phi$
  This is called negative introspection. It can be rewritten as $\neg\Box\psi \to \Box\neg\Box\psi$. If agent Q does not know something, she knows that she does not know it.
- $\Diamond\top$
  This can be rewritten as $\neg\Box\bot$. Agent Q does not know any contradiction.
- $\Box\phi \to \Diamond\phi$
- $\Box(\phi \to \psi) \land \Box\phi \to \Box\psi$
  This is called logical omniscience. Agent Q knows all the consequences of anything she knows.

We can also engineer logics at the level of Kripke models. For each reading of $\Box$, there is a corresponding reading of the accessibility relation R.

- Necessity:
  According to the definition on Slide 26, ϕ is necessarily true at x if ϕ is true in all worlds y accessible from x in a certain way; but in what way? Intuitively, necessarily ϕ is true if ϕ is true in all possible worlds; so $R(x, y)$ should be interpreted as meaning that y is a possible world according to the information at x.
- Knowledge:
  $R(x, y)$ means y could be the actual world according to agent Q's knowledge at x. In other words, if the actual world is x, then agent Q – who is not omniscient – cannot rule out the possibility of it being y.

Properties of Binary Relations

A binary relation R may be:

- reflexive (ρ): if, for every $x \in W$, we have $R(x, x)$;
- symmetric (σ): if, for every $x, y\ inW$, we have $R(x, y)$ implies $R(y, x)$;
- transitive (τ): if, for every $x, y, z \in W$, we have $R(x, y)$ and $R(y, z)$ imply $R(x, z)$;
- serial: if, for every $x \in W$, there is a $y \in W$ such that $R(x, y)$;
- Euclidean: if, for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$;
- an equivalence relation: if it is reflexive, symmetric, and transitive.

Example

For $\Box\phi$ with the meaning 'agent Q knows ϕ', what are the mathematical properties of the binary relation R?

- Should R be reflexive?
  This would say: x could be the actual world according to Q's knowledge at x, i.e. Q cannot have false knowledge. This is a desirable property for R to have. This property corresponds to the validity of the formula $\Box\phi \to \phi$.
- Should R be transitive?
  This would say: that which is known is known to be known (positive introspection). This property corresponds to the validity of the formula $\Box\phi \to \Box\Box\phi$.

Properties of R corresponding to Some Formula Schemes

historical name	formula scheme	property of R
T	$\Box\phi \rightarrow \phi$	reflexive
B	$\phi \rightarrow \Box\Diamond\phi$	symmetric
D	$\Box\phi \rightarrow \Diamond\phi$	serial
4	$\Box\phi \rightarrow \Box\Box\phi$	transitive
5	$\Diamond\phi \rightarrow \Box\Diamond\phi$	Euclidean

Axiomatic Method for Modal Logic

- The axiomatic method allows us to define a class of wff without any reference to their meanings.
- An axiomatic basis for a logical system consists of
  - **(a)** a specification of the language in which the formulae of the system will be expressed, i.e., a list of primitive symbols, together with any definitions that may be thought convenient, together with a set of formation rules specifying which strings of symbols are to count as wff;
  - **(b)** a selected set of wff, known as axioms;
  - **(c)** a set of transformation rules, permitting various operations on the axioms, and also on wff obtained from the axioms by previous applications of the transformation rules.

Axiomatic Method for Modal Logic

- The wff obtained from the axioms, together with the axioms themselves, are known as the theorems of the system.
- If two axiomatic systems, S_1 and S_2, have different bases but contain exactly the same theorems, we shall say that S_1 and S_2 are equivalent.
- If S_1 contains all the theorems of S_2 and other theorems as well, we say that S_1 is a proper extension of S_2, and that S_1 is the stronger and S_2 the weaker of the two systems.

Some important modal logics

We study a few important modal logics that extend the basic modal logic with a consistent set of formula schemes $\mathcal{L}$ which is chosen according to the application at hand.

- The modal logic K
  It is the weakest modal logic, with $\mathcal{L} = \emptyset$. It satisfies all instances of the formula scheme K. Modal logics with this property are called normal.

Definition

A normal modal logic Λ is a set of formulas that contains all PC (Propositional Calculus) tautologies,

(K) $\Box(p \to q) \to (\Box p \to \Box q)$, and

(Dual) $\Diamond p \leftrightarrow \neg\Box\neg p$,

and that is closed under modus ponens, uniform substitution and generalization. We call the smallest normal modal logic K.

The rules of proof (transformation rules) of K are:

- Modus ponens (Rule of Detachment)[MP]: given ϕ and $\phi \to \psi$, prove ψ.
- Uniform substitution [US]: given ϕ, prove θ, where θ is obtained from ϕ by uniformly replacing propositional symbols in ϕ by arbitrary formulas.
- Generalization (Rule of Necessitation)[N]: given ϕ, prove $\Box\phi$.

Example

$\vdash_K \Box(p \wedge q) \to (\Box p \wedge \Box q)$

Proof:

PC	(1)	$(p \wedge q) \to p$
(1) × N	(2)	$\Box((p \wedge q) \to p)$
K	(3)	$\Box(p \to q) \to (\Box p \to \Box q)$
(3) × US		
$(p \rightsquigarrow p \wedge q, q \rightsquigarrow p)$	(4)	$\Box((p \wedge q) \to p) \to (\Box(p \wedge q) \to \Box p)$
(2), (4) × MP	(5)	$\Box(p \wedge q) \to \Box p$
PC	(6)	$(p \wedge q) \to q$
(6) × N	(7)	$\Box((p \wedge q) \to q)$
(3) × US		
$(p \rightsquigarrow p \wedge q)$	(8)	$\Box((p \wedge q) \to q) \to (\Box(p \wedge q) \to \Box q)$
(7), (8) × MP	(9)	$\Box(p \wedge q) \to \Box q$

Example (cont.)

PC	(10)	$(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$
(10) $\times$ US $(p \rightsquigarrow \Box(p \wedge q),$ $q \rightsquigarrow \Box p, r \rightsquigarrow \Box q)$	(11)	$(\Box(p \wedge q) \to \Box p) \to ((\Box(p \wedge q) \to \Box q)$ $\to (\Box(p \wedge q) \to (\Box p \wedge \Box q)))$
(5), (11) $\times$ MP	(12)	$(\Box(p \wedge q) \to \Box q) \to (\Box(p \wedge q)$ $\to (\Box p \wedge \Box q))$
(9), (12) $\times$ MP	(13)	$\Box(p \wedge q) \to (\Box p \wedge \Box q)$

QED

Exercise

$\vdash_K (\Box p \land \Box q) \rightarrow \Box(p \land q)$

Proof:

???

The modal logic KT45

- It extends the modal logic K with $\mathcal{L} = \{T, 4, 5\}$.
- It is used to reason about knowledge.
- T. Truth: the agent Q knows only true things.
- 4. Positive introspection: if the agent Q knows something, then she knows that she knows it.
- 5. Negative introspection: if the agent Q does not know something, then she knows that she does not know it.
- Note that these properties represent idealisations of knowledge. Human knowledge has none of these. Computer agents may have them.